

On the Eddington limit in accretion discs

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ABSTRACT

Although the Eddington limit has originally been derived for stars, recently its relevance for the evolution of accretion discs has been realized. We discuss the question whether the classical Eddington limit – which has been applied globally for almost all calculations on accretion discs – is a good approximation if applied locally in the disc. For this purpose, a critical accretion rate corresponding to this type of modified classical Eddington limit is calculated from thin α -disc models and slim disc models. We account for the non-spherical symmetry of the disc models by computing the local upper limits on the accretion rate from vertical and radial force equilibria separately.

It is shown that the results can differ considerably from the classical (global) value: The vertical radiation force limits the maximum accretion rate in the inner disc region to much less than the classical Eddington value in thin α -discs, while it allows for significantly higher accretion rates in slim discs. We discuss the implications of these results for the evolution of accretion discs and their central objects.

Key words: accretion – Eddington limit – critical disc

1 INTRODUCTION

Similar to the stellar case, disc accretion may be limited by radiation pressure, counteracting gravity and viscous dissipation. The major difference to the stellar case is the different geometry which potentially complicates the situation considerably. While in the stellar case, we are dealing with a – for all practical purposes – spherically symmetric, i.e., 1-dimensional situation, discs require an – at least – 2-dimensional treatment. As yet, however, usually a quasi-stellar Eddington limit is being applied for disc models: (1) spherical symmetry of the system; (2) isotropic radiation; (3) homogeneous degree of ionisation; and (4) no time dependence (stationarity). All these approximations do not apply for the disc case, though it is not clear *a priori* to what degree a proper treatment will alter the resulting numbers. In addition, independent of whether we are dealing with the stellar or the disc case, classically, the Eddington limit uses several approximations which may or may not be justified in all cases: (i) Thomson scattering as the sole source of opacity; (ii) negligible gas pressure, in comparison with radiative pressure; and (iii) no relativistic effects. It is the goal of the present investigation to find out the importance of relinquishing some or all of these approximations.

Over the at least two decades, a number of efforts have been carried out to investigate the ap-

plicability of an Eddington-type limit to the disc case: Jaroszyński, Abramowicz & Paczyński (1980) and Abramowicz, Calvani & Nobili (1980), for instance, have been working on supercritical accretion discs. As a main result of their work, models for so-called slim and thick accretion discs have been developed. In addition to standard radiative cooling, in these discs also cooling by advective flows is taken into account. This permits the critical accretion rate to increase without rising the radiative flux accordingly. The photon trapping mechanism (Begelman 1978; Ohsuga et al. 2002) amplifies this effect even more.

Collin & Kawaguchi (2004) inspected Narrow Line Seyfert Galaxies 1 and concluded that these accrete at super-Eddington rates, while their luminosity stays of the order of the classical Eddington limit. The authors proposed an additional non-viscous energy release in the gravitationally unstable region of the disc, which emits a fraction of the optical luminosity. Recently, Turner et al. (2005) investigated the effect of photon bubble instabilities by two- and three-dimensional numerical radiation MHD calculations, finding that photon bubbles may be important in cooling radiation-dominated accretion discs and therefore confirming previous work by Gammie (1998), Begelman (2002) and Ruszkowski & Begelman (2003). In a similar approach, Meyer (2005) proposed that in strong magnetic fields under radiation pressure, discs fragment into individual magnetically aligned columns of cool disc gas and that radiation

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escapes through the gaps between them at super-Eddington luminosity.

So far most investigations assumed a *global* limit in analogy to the classical Eddington limit to be valid also in the context of an accretion disc. However, Fukue (2000, 2004) questioned this assumption for the first time quantitatively and calculated a *local* Eddington-type limit for radiation pressure dominated geometrically thin discs as well as for slim discs by applying self-similar solutions, based on the work of Watarai & Fukue (1999). He found that inside a certain *critical* radius s_{cr} , the disc remains in an Eddington-critical state with decreasing local accretion rates $\dot{M} = \dot{M}_{\text{input}} \cdot (s/s_{\text{cr}})$, independent of the disc model. The critical radius scales with the mass input rate, $s_{\text{cr}} \approx 1.95(\dot{M}_{\text{input}}/\dot{M}_{\text{E}}) \cdot s_{\text{g}}$, with \dot{M}_{E} being the classical Eddington rate¹ and s_{g} the gravitational radius of the central black hole.

In the present paper, we calculate the analog to the Eddington limit (resp., the critical accretion rate) for a classical thin disc model of the Shakura & Sunyaev (1973) variety and for a slim disc model of the Abramowicz et al. (1988) type. In both descriptions, a standard α -viscosity is applied, therefore restricting the models to the non-selfgravitating case. We relinquish the above approximations (1), (2), (3), (i), and (ii), but keep the approximation of non-relativistic stationarity.

In Sect. 2, we provide the models and derive the equations for the critical accretion rates for both disc types and directions (vertical, radial) separately. In Sect. 3, the results are presented and compared to the classical Eddington accretion rate. Based upon that, a locally Eddington limited disc model is proposed. Section 4 is dedicated to the discussion of our results, and Sect. 5 to an outlook on the possible effects on the evolution of accretion discs.

2 MODELS

The original physical reasoning for the Eddington limit is an equilibrium of the gravitational force F_{g} and the radiative force F_{r} . Contrary to a spherically symmetric body like a star, an accretion disc has to be treated as at least a two-dimensional object. Therefore, one can define two different “Eddington limits”, corresponding to the equilibria in the vertical (z) and radial (equatorial, s) direction:

$$F_{\text{tot}}^{(z)}(s) = F_{\text{r}}^{(z)}(s) - F_{\text{g}}^{(z)}(s) = 0$$

$$F_{\text{tot}}^{(s)}(s) = F_{\text{r}}^{(s)}(s) - F_{\text{g}}^{(s)}(s) + \dots = 0$$

For a rotating viscous disc, further contributions to the total force have to be included. Moreover, the disc Eddington limit is no longer independent of the distance from the center of the object, as is true for the stellar case, leading to *local*, position-dependent Eddington limits.

This section is divided into three parts: The first part introduces the general set-up which applies to thin and slim discs likewise, while the second and third part focus on thin and slim disc models, respectively.

¹ The classical Eddington accretion rate \dot{M}_{E} is derived by equating the disc luminosity with the Eddington limit and deriving a mass flow rate from it (see Sect. 2.1)

2.1 General set-up

In this contribution, we use a cylindrical coordinate system $\{s, \varphi, z\}$ with the distance r to the origin, $r^2 = s^2 + z^2$.

The critical accretion rate at a given position s is denoted by $\dot{M}_{\text{crit}} = \dot{M}_{\text{crit}}(s)$, while the classical Eddington accretion rate is identified as \dot{M}_{E} .

If one naively identifies the classical Eddington luminosity

$$L_{\text{E}} = 4\pi c \frac{GM}{\kappa_{\text{es}}} \quad (1)$$

($\kappa_{\text{es}} = 0.4 \text{ cm}^2 \text{ g}^{-1}$ for Thomson-scattering) with the total energy output of an accretion disc of

$$L_{\text{E}} = \eta \cdot \frac{GM\dot{M}_{\text{E}}}{s_{\text{i}}}, \quad (2)$$

one may derive a quasi-classical Eddington accretion rate

$$\dot{M}_{\text{E}} = \frac{4}{\eta} \cdot \pi c \frac{s_{\text{i}}}{\kappa_{\text{es}}}. \quad (3)$$

Here, s_{i} denotes the disc’s inner radius; the factor η takes care of the coupling between the innermost disc regions and the accreting object. In this paper, we use $\eta = 1/2$. As long as η is of order unity, our conclusions do not depend on its exact value.

We assume an optically thick situation for all our calculations. This, together with the assumption that selfgravity plays no role, has to be justified afterwards.

In contrast to the classical approach, we allow for opacity sources other than Thomson scattering and use an analytic interpolation formula (Gail, priv. comm.; for a similar approach, see also Bell & Lin (1994)):

$$\begin{aligned} \frac{1}{\kappa_{\text{in}}} &= \left[\frac{1}{\kappa_{\text{ice}}^4} + \frac{(3000 \text{ K})^{10}}{(3000 \text{ K})^{10} + T^{10}} \cdot \frac{1}{\kappa_{\text{ice, evap}}^4 + \kappa_{\text{dust}}^4} \right]^{1/4} \\ &+ \left[\frac{1}{\kappa_{\text{dust, evap}}^4 + \kappa_{\text{mol}}^4 + \kappa_{\text{H}}^4} + \frac{1}{\kappa_{\text{atom}}^4 + \kappa_{\text{e}^-}^4} \right]^{1/4} \end{aligned} \quad (4)$$

The individual contributors κ_i are approximated by

$$\kappa_i = \kappa_{0,i} \cdot T^{\kappa_{T,i}} \cdot \rho^{\kappa_{\rho,i}} \quad (5)$$

and are compiled in Table 1.² However, κ_{in} is only meant as a handy interpolation. In particular in the transition regions between individual contributions, one has to be aware of this and should refrain from a physical over-interpretation there.

We also take into account gas pressure, giving the total pressure as

$$p_{\text{tot}} = p_{\text{gas}} + p_{\text{rad}} = \frac{2\rho k_{\text{B}}T}{m_{\text{H}}} + \frac{4\sigma T^4}{3c}. \quad (6)$$

A parameter β describes the contribution of gas to the total pressure in a standard way:

$$\beta = \frac{p_{\text{gas}}}{p_{\text{tot}}} \implies \frac{1-\beta}{\beta} = \frac{2\sigma T^3 m_{\text{H}}}{3c\rho k_{\text{B}}} \quad (7)$$

² Slight differences, for instance for $\kappa_{\text{ice, evap}}$ by a factor of 2, between Bell & Lin (1994, Table 3) and our values, are due to somewhat different ways of combining the individual contributors. For the purpose of the present investigation, these differences are of minor importance

Table 1. Interpolation of the opacity: set of parameters (in cgs-units)

Contributor <i>i</i>	Symbol	$\kappa_{i,0}$	$\kappa_{i,\rho}$	$\kappa_{i,T}$
Dust with ice mantles	κ_{ice}	$2.0 \cdot 10^{-4}$	0	2
Evaporation of ice	$\kappa_{\text{ice, evap}}$	$1.0 \cdot 10^{16}$	0	-7
Dust particles	κ_{dust}	$1.0 \cdot 10^{-1}$	0	1/2
Evaporation of dust particles	$\kappa_{\text{dust, evap}}$	$2.0 \cdot 10^{81}$	1	-24
Molecules	κ_{mol}	$1.0 \cdot 10^{-8}$	2/3	3
Negative hydrogen ion	κ_{H^-}	$1.0 \cdot 10^{-36}$	1/3	10
Bound-free and free-free-transitions	κ_{atom}	$1.5 \cdot 10^{20}$	1	-5/2
Electron scattering	κ_{e^-}	0.348	0	0

A main point of interest is the total luminosity of an Eddington limited accretion disc. We calculate the disc's luminosity by

$$L_{\text{disc}} = 2 \cdot 2\pi \cdot \int_{s_i}^{s_o} ds \sigma T_{\text{eff}}^4, \quad (8)$$

where T_{eff} stands for the effective temperature (surface temperature) of the accretion disc at radius s , which is related to the central temperature T_c by the optical depth $\tau = \kappa \rho h$,

$$\sigma T_{\text{eff}}^4 = \frac{16}{3\tau} \sigma T_c^4, \quad (9)$$

replacing the vertical stratification by a one-zone approximation.

Since our models take advantage of the α -description for the viscosity, the mass of the disc has to be small compared to the mass M of the central object (Duschl, Strittmatter & Biermann 2000). For later comparison with the central mass, we introduce the disc's mass

$$M_{\text{disc}} = \int_{s_i}^{s_o} \Sigma 2\pi s ds \quad (10)$$

with the surface density $\Sigma = 2\rho h$. The height h of the disc is defined by the vertical hydrostatic equilibrium, $p/h = \rho g_z$, where the mass density ρ and the pressure p are measured in the disc plane ($z = 0$) and the gravitational acceleration g_z is taken from the disc surface ($z = h$).

2.2 Geometrically thin accretion discs

Geometrically thin models have been the most popular description of accretion discs in the last 30 years. Based upon the assumption of a negligible height h of the disc relative to the radius r and neglecting advective cooling, this model is simple but powerful for many applications. We derive the critical accretion rate separately for the vertical and radial direction by assuming a balance of force between the out-dragging radiative force F_r and the attracting forces, like the gravitational force F_g .

2.2.1 Vertical direction

In this case, we consider a particle with mass m_H at the surface h of the disc. This particle is attracted by the vertical component of the gravitational force

$$F_g^{(z)} = \frac{GMm_H}{s^2} \cdot \frac{h}{s}, \quad (11)$$

while it is accelerated outwards by the radiative force

$$F_r^{(z)} = \frac{\kappa m_H}{c} \sigma T_{\text{eff}}^4, \quad (12)$$

according to the Stefan-Boltzmann law. Using

$$\sigma T_{\text{eff}}^4 = \frac{3}{8\pi} \frac{GM\dot{M}}{s^3} f \quad (13)$$

and equating $F_g^{(z)}$ and $F_r^{(z)}$, we obtain an equation for the *local* critical accretion rate:

$$\dot{M}_{\text{crit}}(s) = \frac{8\pi}{3} \frac{c}{\kappa} \cdot \frac{h}{f} \quad (14)$$

Here, f is a function describing the radially inner boundary condition. A torque-free condition, for instance, leads to $f = 1 - \sqrt{s_i/s}$.

Equation (14) changes the quality of the radial mass flow rate. While in standard disc theory, it is a parameter, it now becomes a local solution of our criticality assumption.

By means of the remaining disc equations and (7), we solve this set of equations for β and finally end up with $\beta = 3/4$, independent of a specific choice of the opacity κ . This leads to

$$\dot{M}_{\text{crit}}(s) = \frac{1}{f} \cdot \frac{8\pi}{3} \cdot \frac{8^{3/8}}{3^{1/2}} \cdot \frac{k_B^{1/2} c^{5/4}}{m_H^{1/2} \alpha^{1/8} \kappa^{9/8} \sigma^{1/8} \Omega_K^{7/8}}, \quad (15)$$

where Ω_K stands for the Keplerian angular velocity. Contrary to the solution for β , the critical accretion rate depends on the opacity.

2.2.2 Radial direction

The calculation of the critical accretion rate given by a balance of radial forces is less straight-forward then for the vertical direction. We consider a particle in the equatorial plane of the disc, at $z = 0$. So, the attracting gravitational force is given by

$$F_g^{(s)} = \frac{GMm_H}{s^2}. \quad (16)$$

In contrast to our previous derivation, the particle is located inside the disc and no longer at its surface. Assuming our particle being located at a specific position s in an optically thick medium, additionally to the out-dragging radiation originating from the inner shell at $s - ds$ we have to take into account also the radiation of the outer shell at $s + ds$ pointing inwards. In brief, we have to apply the radial difference of the radiative force instead of its absolute value. Furthermore, we have to consider the centrifugal force due to azimuthal motion and, coupled with it, the viscous forces.

Putting all together, this leads us to the radial component of the Navier-Stokes equation, which is given by

$$F_{\text{tot}}^{(s)} = m_H \Omega_K^2 s - \frac{m_H}{\rho} \cdot \frac{\partial p_{\text{tot}}}{\partial s} - \frac{m_H}{2} \frac{\partial v_s^2}{\partial s} + m_H \nu \frac{\partial^2 v_s}{\partial s^2} - m_H \nu \frac{v_s}{s^2} - \frac{GM m_H}{s^2} = 0. \quad (17)$$

By means of the disc equations, the condition of a vanishing total force yields a differential equation for β . In the Keplerian case, the centrifugal and gravitational force (i.e., the first and the last term of (17)) cancel each other, leaving the remainder of (17) to be solved.

Independent of the chosen boundary values, its solution leads to supersonic radial motion, $|v_s| > c_s = (p_{\text{tot}}/\rho)^{1/2}$, which contradicts the thin disc model. Hence, we investigate the corresponding value of the accretion rate under the condition $v_s = A c_s$, $|A| < 1$, which can subsequently be expressed as

$$\dot{M}_{\text{crit}}(s) = \frac{32\pi c}{3\kappa} \cdot \frac{A}{\alpha} \cdot \frac{s}{(1-\beta)}. \quad (18)$$

2.3 Slim accretion discs

Abramowicz et al. (1988) extended the model of supercritical accretion discs by Jaroszyński et al. (1980) to slim accretion discs with $h \lesssim s$. Since the height h is no longer negligible compared to the equatorial distance s , the radial momentum equation has to be solved in a more precise way, leading to a non-Keplerian rotation of the disc. Furthermore, beside of the standard cooling by radiation, also cooling by advective mass flows becomes important, requiring a more detailed treatment of the energy (transport) equation.

As the vertical disc structure is known only approximately, the authors used three correcting factors (B_1 , B_2 and B_3), which are all of the order unity, to correct the results of the vertical integration.³

- B_1 takes into account that the height of the disc is no longer negligible compared to its radius, $(s^2 + h^2)^{3/2} \approx B_1 \cdot s^3$. In the model, we use a Newtonian potential; combined with the assumption of hydrostatic equilibrium, this leads to

$$\frac{p_{\text{tot}}}{\rho} = \frac{GM \cdot h^2}{(s^2 + h^2)^{3/2}} \approx \frac{GM \cdot h^2}{B_1 \cdot s^3} = \frac{\Omega_K^2 h^2}{B_1}. \quad (19)$$

- B_2 is a measure of how efficiently the viscously dissipated energy is converted into radiation (rather than radially advected),

$$F_r^{(z)} = B_2 \frac{m_H}{\rho} p_{\text{rad}} h. \quad (20)$$

- B_3 measures the efficiency of the advective energy transport in the energy transport equation and as such is part of modeling the advective processes.

³ These parameters were introduced originally by Paczyński & Bisnovatyi-Kogan (1981). However, the notation of the parameters differs slightly between these two publications. In this paper, we follow Abramowicz et al. (1988)

2.3.1 Vertical direction

The resulting set of disc equations is given in Abramowicz et al. (1988, p. 648). Starting from there and using the parameter β from (7), we derive two equations for the three unknowns β , h and l , where l is the local specific angular momentum. The missing third equation is again given by the Eddington condition of a vanishing total force:

$$F_{\text{tot}}^{(z)} = F_r^{(z)} + F_g^{(z)} = 0 \quad (21)$$

With

$$F_g^{(z)} = \frac{GM m_H}{(s^2 + h^2)} \cdot \frac{h}{\sqrt{s^2 + h^2}} \approx \frac{GM m_H h}{B_1 s^3} \quad (22)$$

in terms of the slim disc description, and

$$\begin{aligned} F_r^{(z)} &= B_2 \frac{m_H}{\rho} p_{\text{rad}} h = (1-\beta) B_2 \frac{m_H p_{\text{tot}}}{h \rho} \\ &= (1-\beta) \frac{B_2}{B_1} \frac{GM m_H h}{s^3}, \end{aligned} \quad (23)$$

the Eddington condition becomes

$$\beta = 1 - \frac{1}{B_2}. \quad (24)$$

Also in this model, the ratio of gas pressure to total pressure remains constant and is determined by the choice of the parameter B_2 . The critical accretion rate as well as the other disc quantities can be calculated by solving the two (coupled) differential equations numerically (Appendix A).

2.3.2 Radial direction

We find that similar to the thin disc case presented in Sect. 2.2.2, the radial Eddington limit in slim accretion discs leads to supersonic radial velocities, $|v_s| > c_s$, which cannot be the case for (stationary) slim discs. Therefore, we follow the same idea as in Sect. 2.2.2 and require $v_s = A c_s$ with $|A| < 1$. In terms of the slim disc model, we obtain

$$v_s = \frac{s \alpha \Omega_K^2 h^2}{B_1 B_2 (l - l_0)}. \quad (25)$$

The condition $v_s = A c_s$ leads to

$$h = B_2 \sqrt{B_1} \cdot \frac{A}{\alpha} \cdot \left(\frac{l}{l_K} - \frac{l_0}{l_K} \right) s. \quad (26)$$

Here, l_K and l_0 are the local Keplerian specific angular momentum and the specific angular momentum of the disc's innermost orbit, respectively.

Starting from the radial momentum equation and the energy equation for slim discs, we can derive two additional differential equations for the unknowns h , β and l . Together with (26), the system is closed and can be solved numerically. The set of equations is shown in appendix B.

3 RESULTS

As can be seen from the above equations, the critical accretion rates depend on the distance s from the central object. We wish to emphasize that these accretion rates represent the *local* maximum value of mass which can be transferred inwards at a specific distance from the center in a stationary accretion state.

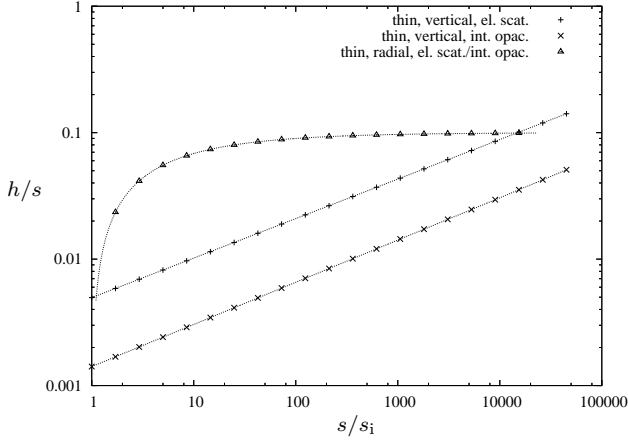


Figure 1. Relative height h/s for the vertical Eddington limit and for the radial limit in standard α -discs

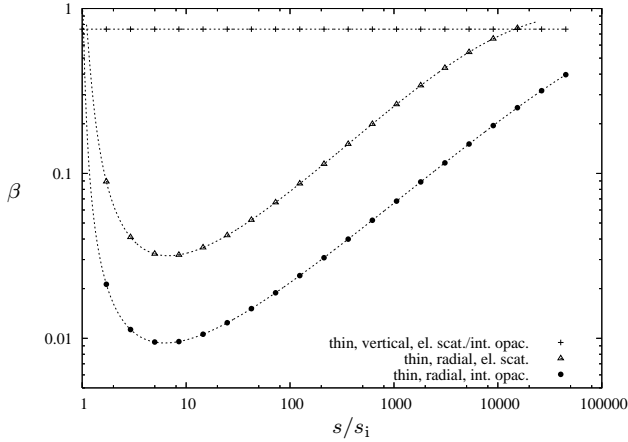


Figure 2. $\beta = p_{\text{gas}}/p_{\text{tot}}$ for the vertical Eddington limit and for the radial limit in standard α -discs

For the following discussion of our results, we define a *reference model*: The central object is assumed to be a stellar black hole with $M = 10M_{\odot}$, resulting in an inner radius of $s_i = 6GM/c^2 = 8.86 \cdot 10^6$ cm. The outer radius s_o is set to $5 \cdot 10^4 s_i$. In this environment, the classical Eddington rate is $\dot{M}_E = 1,67 \cdot 10^{19} \text{ g s}^{-1} = 2.65 \cdot 10^{-7} M_{\odot} \text{ a}^{-1}$. The viscosity parameter α is set to 0.1. A torque-free boundary condition is chosen at the inner disc radius $s = s_i$: $f = 1 - \sqrt{s_i/s}$.

In the following figures, the curves are labeled with the corresponding disc model (**thin**, **slim**), local Eddington limit (**vertical**, **radial**) and opacity (**el. scat.**: $\kappa = \kappa_{\text{es}}$, **int. opac.**: $\kappa = \kappa_{\text{in}}$).

3.1 Thin accretion discs

Figures 1–3 present the results for the vertical and the radial direction – in the latter case, the parameter A is set to 0.01, corresponding to $h/s \approx 0.1$ as an upper limit for thin discs. This choice is somewhat arbitrary, but provides a reasonable upper limit for the radial velocities in thin discs. According to (18), the allowed mass accretion rate scales linearly with the parameter A .

Figure 1 shows that the height of Eddington limited

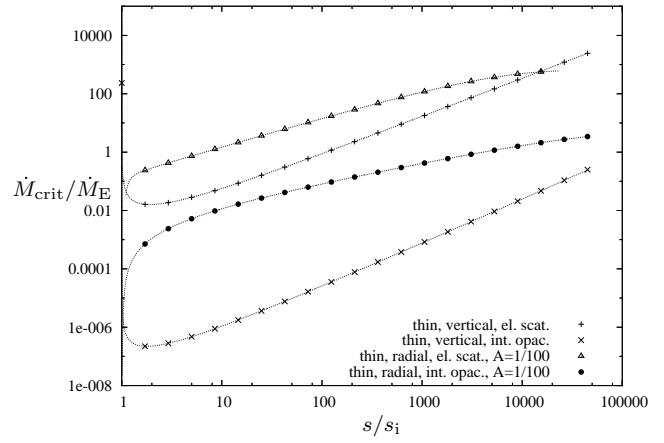


Figure 3. Critical accretion rate \dot{M}_{crit} in units of the classical Eddington rate \dot{M}_E for the vertical Eddington limit and for the radial limit in standard α -discs

discs decreases steadily with the radius s , while it remains constant for a large range in the radial limit. This is a direct consequence of the gas-to-total-pressure ratio β , which is constant only for the vertical Eddington limit (Fig. 2). In the radial limit, β decreases with s by two orders of magnitude, causing the radiation pressure to exceed the gas pressure and to expand the disc vertically. This leads to higher surface densities and radial velocities, and therefore to higher critical accretion rates, as it can be seen from Fig. 3. Furthermore, we observe a notable influence of the opacity on the results: For κ_{es} , the critical accretion rates are much higher than for κ_{in} , reaching a value of approximately the classical Eddington rate \dot{M}_E in the inner part of the disc. According to (12) and the fact that in thin discs all energy released by viscous processes is transported by radiation, the radiation force depends linearly on κ , and $\dot{M}_{\text{crit}} \propto \kappa^{-1}$, see (14).

For some parameter ranges we find rather large differences between the Eddington limits derived following the standard approach of taking into account electron scattering as the only absorbing process (κ_{es}), and those which allow for additional absorption processes (κ_{in}). In the present calculations, bound-free and free-free processes (contributor κ_{atom} in (4)) dominate the opacity κ_{in} and alter it significantly. The resulting differences in the Eddington limits are of relevance and emphasize the importance of taking into account absorption processes other than Thomson scattering in the disc case, i. e., of using consistent opacities.

The changes in the inner few s_i is due to the boundary condition f at the inner radius: For $s \rightarrow s_i$ the boundary function f tends to zero. As \dot{M}_{crit} scales with f^{-1} (see (15)), the critical accretion rate formally diverges. This, however, has to be taken as a technical artefact: In reality, the stress at the inner boundary will always be nonzero, leading to finite values for \dot{M}_{crit} . However, f may be still be small and therefore theoretically allowing for higher accretion rates than at the minimum at $s_{\text{min}} \approx 2s_i$. As long as no vertical mass inflows to the disc are present, the *real* accretion rate \dot{M} will stay at its minimum value $\dot{M}_{\text{crit}}(s_{\text{min}})$ and the disc will be in a sub-Eddington state in terms of our local Eddington-limit for $s_i < s < s_{\text{min}}$.

For the vertical Eddington limit and also for the radial limit, the condition of a non-selfgravitating disc holds,

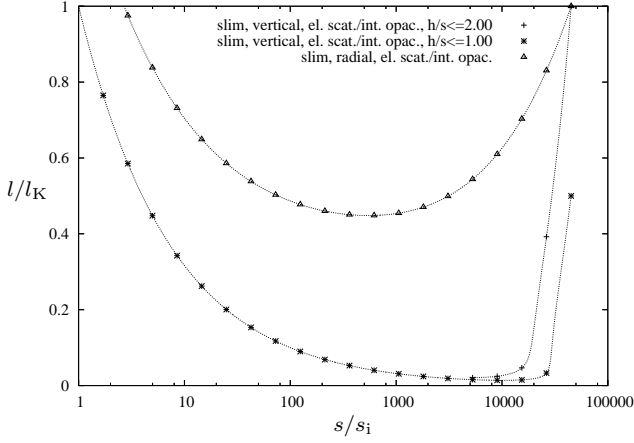


Figure 4. Angular momentum l in units of the Keplerian angular momentum l_K for the vertical Eddington limit and for the radial limit in slim discs

$M_{\text{disc}} \ll M$. Also the assumption of optical thickness is fulfilled for both κ_{es} and κ_{in} ,

$$\int_0^h \rho \kappa dz \approx \frac{1}{2} \Sigma \kappa > 1.$$

3.2 Slim accretion discs

For the model calculations, we choose the three parameters B_1 , B_2 , and B_3 as follows:

- The slim disc model is valid for $h \lesssim s$, i. e. $(s^2 + h^2)^{3/2} = B_1 s^3 \lesssim 2^{3/2} s^3$, making $B_1 = 2$ a reasonable choice.
- For the vertical Eddington limit, $\beta = 1 - 1/B_2$. In order to compare our results directly to thin accretion discs with $\beta = 3/4$, we set $B_2 = 4$.
- We follow Paczyński & Bisnovatyi-Kogan (1981) and set $B_3 = 1/2$.

In order to solve the sets of differential equations, we have to define boundary conditions. Due to numerical reasons, this has to be done at the outer boundary. We set $l(s_o) = l_K(s_o)$ and $\beta(s_o) = 3/4$ for the radial limit. With $A = 1/30$ (for slim discs, radial velocities are usually higher than for thin discs) and (26), this leads to $h(s_o) = 2s_o$. Also here, the choice of A is somewhat arbitrary. Since the dependency of the allowed mass accretion rates on A cannot be seen directly from the equations, we also compute the case $A = 1/20$. For the vertical limit, we set $l(s_o) = l_K(s_o)$ and $h(s_o) = 2s_o$. For comparison, we also compute the vertical limit with $h(s_o) = s_o$. The results are shown in Figs. 4–7.

As it can be seen from Fig. 4, the angular momentum drops with respect to the local Keplerian value $l_K(s)$ for both the radial and the vertical limit. For the latter one, the decrease is extremely steep, resulting in $l(s) = (1 + \varepsilon)l_o$, $\varepsilon \ll 1$. For $s \rightarrow s_i$, the angular momentum reaches its Keplerian value in the vertical limit, while it overshoots this limit at $s \approx s_i$ in the radial limit.

Figure 5 shows the behaviour of the disc's relative height h/s for local equilibria of forces in the vertical and radial direction, respectively. While this ratio scales with s in the vertical limit, it adopts a constant value about unity

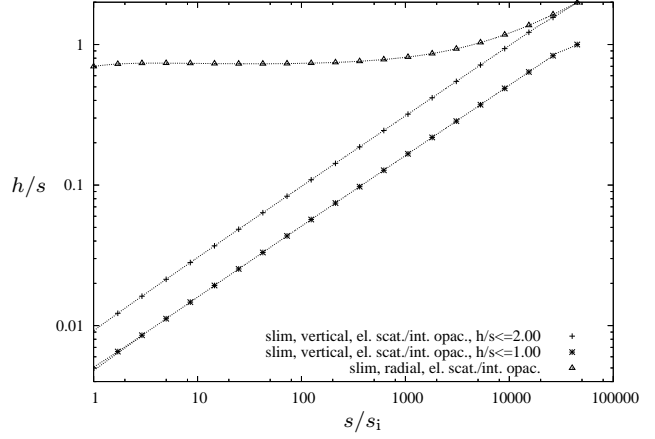


Figure 5. Relative height h/s for the vertical Eddington limit and for the radial limit in slim discs

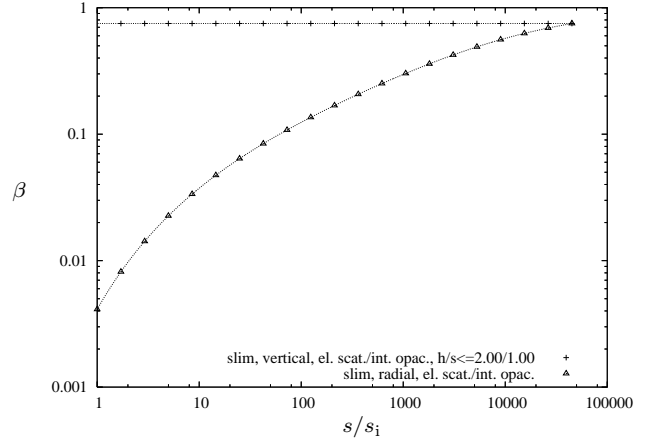


Figure 6. $\beta = p_{\text{gas}}/p_{\text{tot}}$ for the vertical Eddington limit and for the radial limit in slim discs

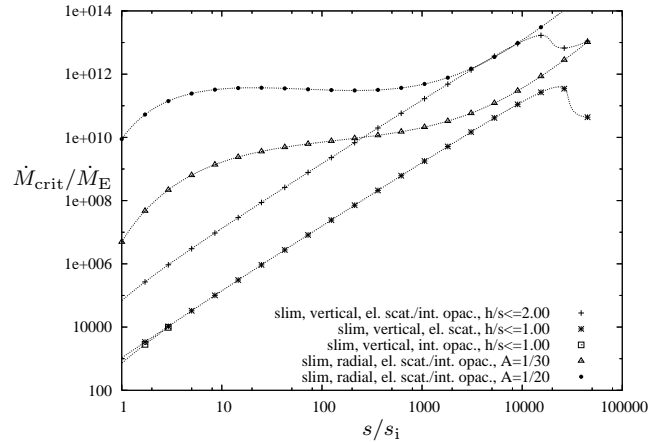


Figure 7. Critical accretion rate \dot{M}_{crit} in units of the classical Eddington rate \dot{M}_E for the vertical Eddington limit and for the radial limit in slim discs

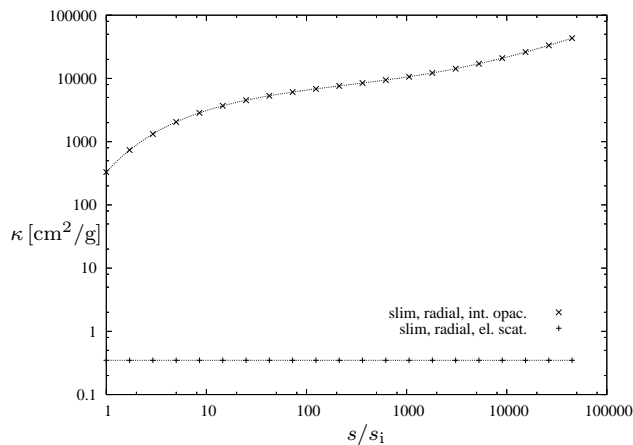


Figure 8. κ_{es} and κ_{in} for slim discs in the radial limit

in the radial limit. In analogy to thin discs, this corresponds to a decrease of β in the latter case for smaller radii (Fig. 6) and to higher accretion rates with respect to the vertical limit (Fig. 7). The critical accretion rates exceed the limits for thin discs (cf. Sect. 3.1) significantly.

We want to point out that the results are almost independent of the opacity, although the absolute values of κ differ considerably (Fig. 8). This becomes plausible when regarding the energy equation (Abramowicz et al. 1988, (12)): For high accretion rates, the radiative processes are negligible compared to the advective flows.

In all cases, the assumption of large optical thickness is fulfilled. By computing realistic disc models (see Sect. 3.3 for a more detailed discussion), we find that also the condition of a negligible disc mass is fulfilled as long as the mass input rate at the outer boundary $\dot{M}(s_o)$ stays at a reasonable value below $10^5 \dot{M}_E$.

3.3 An Eddington limited thin disc

We have seen in Sect. 3.1 that the Eddington limit poses strong limitations on the mass inflow rate in classical thin accretion discs. Contrary to the above sections where we displayed the disc properties for critical accretion rates at every radius, we now construct a stationary thin disc model by taking into account the local Eddington limit only where needed.

Let us assume an external mass input rate \dot{M}_o at the outer boundary s_o . As the critical accretion rates in the outer part of a thin α -disc become quite large, we expect the disc there to be sub-critical for any reasonable \dot{M}_o . But, at a certain radius s_{crit} – which can be calculated directly from (15) – the external mass input rate may become equal to the local Eddington limit $\dot{M}_{\text{crit}}(s_{\text{crit}})$. As $\dot{M}_{\text{crit}}(s)$ is monotonically decreasing (for the influence of the inner boundary condition, see the discussion above), the disc will remain in a critical state for all $s \leq s_{\text{crit}}$ when assuming that mass outflows guarantee the condition $\dot{M}(s) = \dot{M}_{\text{crit}}(s)$ for $s \leq s_{\text{crit}}$.

This leads to discrepancies in the disc model, as the α -disc equations have been derived under the condition $\dot{M} = \text{const.}$ This requires the application of non-conservative disc models as presented by Lipunova (1999). As can be seen

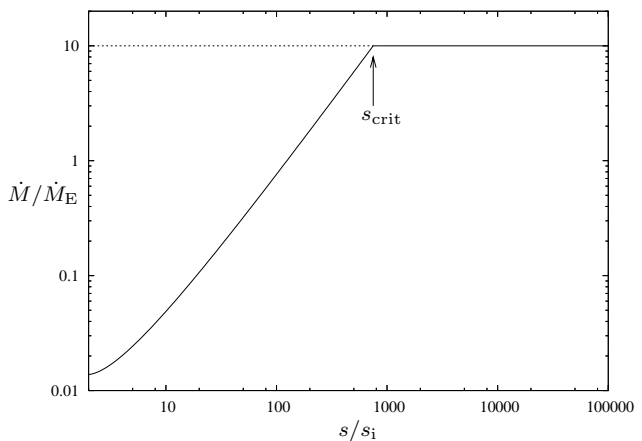


Figure 9. Accretion rate for a thin disc, $\dot{M}_o = 10\dot{M}_E$. Solid/ dashed line: with/without consideration for the local (vertical) Eddington limit

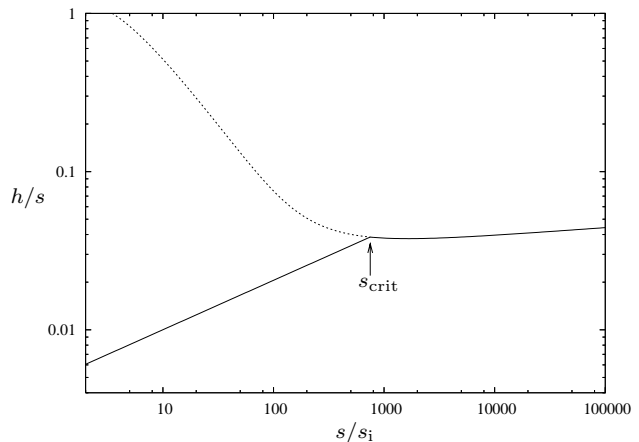


Figure 10. Relative disc height h/s for a thin disc with $\dot{M}_o = 10\dot{M}_E$. Solid/ dashed line: with/without consideration for the local (vertical) Eddington limit

from her results, the disc equations differ only slightly from that of a conservative disc model with constant accretion rate. This is also what we expect from comparing the time scales: The disc will adjust to changes in the accretion rate in about the hydrostatic timescale $t_{\text{hyd}} = h/c_s$, which itself is much shorter than the viscous timescale $t_{\text{vis}} = s/v_s$, as long as the disc remains thin. Therefore, we can approximate an Eddington limited thin disc with a quasi-static model.

Figures 9–11 show the results for such a thin disc with $\dot{M}_o = 10\dot{M}_E$. For simplicity, we adopted simple electron scattering throughout the whole disc and ignored the inner $2s_i$ because of the influence of the boundary condition. In this case, $s_{\text{crit}} \approx 750s_i$. As can be seen from these figures, the mass input rate to the central object drops to about $10^{-2}\dot{M}_E$ in the range $[2s_i; s_{\text{crit}}]$. Also, the ratio h/s follows this trend, in contrast to the pseudo thin disc solution shown in dashed lines with $\dot{M}(s) = \dot{M}_o = \text{const.}$ The disc inflation in the latter conservative case corresponds to a transition into a radiation pressure supported disc and violates the assumptions of geometrical thinness. On the other hand, this condition of negligible relative height of the disc holds for

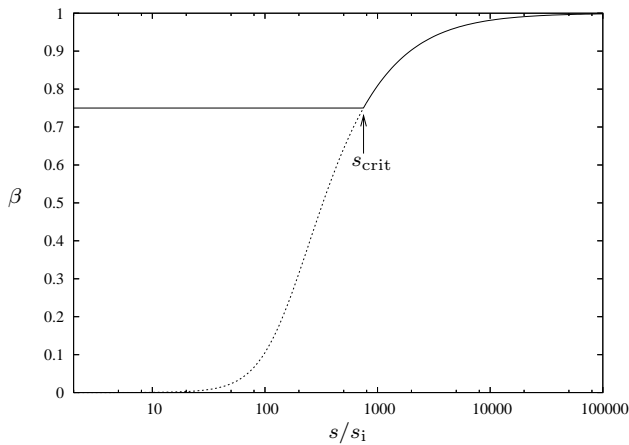


Figure 11. $\beta = p_{\text{gas}}/p_{\text{tot}}$ for a thin disc, $\dot{M}_0 = 10\dot{M}_E$. Solid/dashed line: with/without consideration for the local (vertical) Eddington limit

an Eddington limited thin disc, which is mainly supported by gas pressure.

The total disc luminosity, given by (8), becomes $0.15L_E$ for the Eddington limited case and $10L_E$ for the pseudo thin disc case. The total disc masses (10) almost equal each other, $M_{\text{disc}} \approx 10^{-6}M$ (M : mass of the central object), reflecting the fact of a simple “puffing up” of the pseudo thin disc.

4 DISCUSSION

In order to interpret our results correctly, we have to keep in mind that the disc quantities presented in Sect. 3.1 and 3.2 correspond to discs at their respective *local* vertical and radial Eddington limits. The critical accretion rates in the outer regions are enormous. However, such high mass input rates are only formal solutions as they will never be reached in real systems. The allowed mass accretion rates are decreasing constantly with smaller radii, confirming our expectations that the Eddington limit becomes important only in the inner part of the disc.

In all our models, a strong correlation between the relative height h/s of the disc and the critical accretion rate \dot{M}_{crit} is present. For a constant ratio of gas pressure to total pressure β , the relation h/s scales like $s^{0.6}$. From the analysis of the radial limit, we know that with decreasing β for a constant radius, the disc is blown up in vertical direction, i.e. h/s increases. Associated with this development, but less effective, \dot{M}_{crit} increases too. We have also seen that for sufficiently high \dot{M} and h/s , the angular momentum in the disc differs significantly from its Keplerian value over large parts of the slim disc, allowing advective flows to dominate the energy transport and to rise the critical accretion rates.

We can now draw a more realistic scenario for an accretion disc system: Let us assume a given reasonable mass input rate \dot{M}_0 at the outer boundary of the disc. Doing so, we can certainly assume the disc to be thin in its outer parts. For smaller radii, the ratio h/s will begin to increase as long as the vertical Eddington limit is not violated and therefore no outflows occur, see Fig. 10. Further inwards, two competing effects now come into play: On the one hand, outflows may occur and help the disc to stay in a subcritical state;

on the other hand, the thin disc may pass into a slim disc with deviations from Keplerian rotation, advective energy transport and a changing ratio β . In reality, both effects will be in concurrence. To answer this question, extended time-dependent disc models have to be investigated.

5 CONCLUSION

Our work shows that the quasi-spherical treatment of accretion discs by applying the classical Eddington limit does not hold as a good approximation. The allowed mass accretion rates differ significantly from the stellar case and also depend strongly on the disc model, while the disc luminosity only weakly exceeds the classical Eddington value.

We have seen that the inner domain of an accretion disc holds the key position for the evolution and also for the observational appearance of highly accreting systems: The final growth rate of the central object, the total amount of matter that has to be expelled from the disc and also most its luminosity are determined by it. Numerous radiation hydrodynamic simulations (see Ohsuga et al. (2005) for one example) show that outflows in the middle part of the disc get stuck and may therefore be driven back to the disc, while mass ejections in the inner part lead to high velocities and gravitationally unbound streams of previously accreted material.

It is broadly accepted that photon trapping effects, which are not included in the slim disc model (Ohsuga et al. 2002), alter the emerging luminosity and therefore may change our results in a way that even higher critical accretion rates are possible. Also, the effect of further energy transport mechanisms, like convection and heat conduction, and non-stationarity have to be investigated in a more sophisticated discussion.

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APPENDIX A: VERTICAL EDDINGTON LIMIT IN SLIM ACCRETION DISCS

The parameterization of the pressure allows us to reduce all thermodynamics to the parameter β :

$$T_c = \beta \cdot \frac{m_H \Omega_K^2 h^2}{2k_B B_1} \quad (\text{A1})$$

$$p_{\text{tot}} = \frac{\beta^4}{1-\beta} \cdot \frac{\sigma}{12c} \cdot \frac{m_H^4}{k_B^4} \cdot \frac{\Omega_K^8 h^8}{B_1^4} \quad (\text{A2})$$

$$\rho = \frac{\beta^4}{1-\beta} \cdot \frac{\sigma}{12c} \cdot \frac{m_H^4}{k_B^4} \cdot \frac{\Omega_K^6 h^6}{B_1^3} \quad (\text{A3})$$

We use the result (24), $\beta = \text{const.}$, together with equations (3)–(13) of Abramowicz et al. (1988). More precisely, we substitute all unknowns except of h and l in the radial momentum equation (Abramowicz et al. 1988, (8)) and the energy equation (Abramowicz et al. 1988, (12)) by using the remaining disc equations and the Eddington condition (24). Note that (7) contains an error in the second term in parentheses: The factor $(4 - \beta)$ has to be replaced by $(4 - 3\beta)$ (Jaroszynski et al. 1980).

This leads us to two differential equations for the disc height h and the angular momentum l , namely

$$\frac{8\Omega_K h}{B_1} \cdot \frac{d(\Omega_K h)}{ds} + \left(\frac{l_K^2}{s^3} - \frac{l^2}{s^3} \right) = - \frac{\alpha \Omega_K^3 h^3 s}{B_1 B_2 (l - l_0)^2} \cdot \left\{ \Omega_K h + 2s \frac{d(\Omega_K h)}{ds} - \frac{\Omega_K h s}{(l - l_0)} \frac{dl}{ds} \right\} \quad (\text{A4})$$

and

$$\frac{(1-\beta)^2}{\beta^4} \cdot \frac{12B_2 B_1^3 c^2}{\alpha \sigma \kappa} \cdot \frac{k_B^4}{m_H^4} \cdot \frac{(l - l_0)}{\Omega_K^6 h^8 s} = (l - l_0) \cdot \left(\frac{2l}{s^3} - \frac{1}{s^2} \frac{dl}{ds} \right) - 3\beta \frac{B_3}{B_1} \Omega_K h \frac{d(\Omega_K h)}{ds}. \quad (\text{A5})$$

Herein, l_K (resp. Ω_K) stands for Keplerian azimuthal motion and l_0 is given by the boundary condition of a vanishing viscous torque at the inner boundary s_i of the accretion disc, $l_0 = l_K(s_i)$. The critical accretion rate can be calculated from the solutions of (A4) and (A5) by

$$\begin{aligned} \dot{M}_{\text{crit}}(s) &= \frac{4\pi s^2 \alpha h p_{\text{tot}}}{l - l_0} \\ &= \frac{\beta^4}{1-\beta} \cdot \frac{\alpha \sigma \pi}{3c B_1^4} \cdot \frac{m_H^4}{k_B^4} \cdot \frac{\Omega_K^8 h^9 s^2}{l - l_0}. \end{aligned} \quad (\text{A6})$$

APPENDIX B: RADIAL EDDINGTON LIMIT IN SLIM ACCRETION DISCS

Again, we use the β -description for the disc's thermodynamics, which yields to (A1)–(A3). The missing equation to be able to determine \dot{M}_{crit} from the system is now given by $v_s = A c_s$ instead of the Eddington condition (24). From the disc equations and from the first law of thermodynamics, we already derived (26):

$$h = B_2 \sqrt{B_1} \cdot \frac{A}{\alpha} \cdot \left(\frac{l}{l_K} - \frac{l_0}{l_K} \right) s \quad (\text{26})$$

As we have seen in Sect. 2.2.2, we cannot assume $\beta = \text{const.}$ in this case. The radial momentum equation and the energy equation lead to two differential equations for h , l and β , namely

$$\begin{aligned} \frac{d}{ds} \ln \left(\frac{\beta^4}{(1-\beta)} \right) + \left(8 + \frac{2\alpha^2 s^2 \Omega_K^2 h^2}{B_1 B_2^2 (l - l_0)^2} \right) \cdot \frac{d \ln(\Omega_K h)}{ds} = \\ \frac{B_1 s}{\Omega_K^2 h^2} (\Omega^2 - \Omega_K^2) + \frac{\alpha^2 s^2 \Omega_K^2 h^2}{B_1 B_2^2 (l - l_0)^2} \cdot \frac{(l + l_0 + s^3 \frac{d\Omega}{ds})}{(l - l_0) \cdot s} \end{aligned} \quad (\text{B1})$$

and

$$\begin{aligned} \frac{B_2 B_1^3 12c^2}{\kappa \sigma \Omega_K^6 h^8 s \alpha} \cdot \frac{k_B^4}{m_H^4} \cdot \frac{(1-\beta)^2}{\beta^4} + \frac{d\Omega}{ds} = \\ \frac{B_3 \Omega_K^2 h^2}{B_1 (l - l_0)} \cdot \left\{ \frac{1}{2} \left(\frac{3\beta^2 + 3\beta - 8}{\beta(1-\beta)} \right) \frac{d\beta}{ds} - \frac{3\beta}{\Omega_K h} \frac{d(\Omega_K h)}{ds} \right\}. \end{aligned} \quad (\text{B2})$$

The critical accretion rate is again given by (A6). Together with (26), this system can be solved numerically.